Carryless Arithmetic (I): The Mod 10 Version

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In honor of Martin Gardner (October 21, 1914 – May 22, 2010).

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Abstract What might arithmetic look like on an island that eschews carry digits?

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1. Nim

The game of Nim was described by Martin Gardner in one of his earliest columns (see [7]), and over the years he returned to this game and its generalizations in many subsequent columns [8]–[16]. Forms of the game have been played since antiquity, with a complete theory first published in 1902 [3].

Central to the analysis of Nim is the operation of Nim-addition. The Nim-sum of a set of numbers is formed by writing them in base 2 and adding the columns mod 2, with no carries to the next column. (This is the binary "Exclusive-OR" of their values.) A Nim position is a winning position for a player if and only if the Nim-sum of the sizes of the heaps is zero [2], [7].

It is natural to wonder if there is a generalization of Nim in which the analysis uses the base-b representations of the sizes of the heaps, for b greater than 2, and in which a position is a win if and only if the mod-b sums of the columns is identically zero. One such game, Rim_b (an abbreviation of Restricted-Nim) exists, although it is complicated and not well known. It was introduced in an unpublished paper [6] in 1980 and is hinted at in [5]. Yet despite his interest in Nim, Martin Gardner never mentions Rim_b, nor does it appear in *Winning Ways* [2], which extensively analyzes Nim variants.

In the present paper we focus on the case $b = 10$, and consider, not the game Rim_{10} itself, but the arithmetic that arises if calculations, both addition and multiplication, are performed mod 10, with no carries.

Along the way we will encounter several new and interesting number sequences. These would also have appealed to Martin Gardner, since he was always a fan of integer sequences. When the third author's *Handbook of Integer Sequences* was published 38 years ago, he was kind enough to say in his *Mathematical Games* column for July 1974 that "every recreational mathematician should buy a copy forthwith." That book contained 2372 sequences: today its successor, the *On-Line*

Encyclopedia of Integer Sequences (or OEIS) [17], contains nearly 200,000 sequences. Alas, we were about to write to Martin about carryless arithmetic, a topic we thought he would enjoy, when we heard the sad news of his death. This article[1] is offered in his honor.

2. The Carryless Islands

We set the stage with a story, of the sort sometimes favored by Gardner, which explains how this carryless arithmetic *might* have arisen.

The fabled carefree residents of the Carryless Islands in the remote South Pacific have very few possessions, which is just as well, since their strange notion of arithmetic is ill-suited to accurate bookkeeping. When they add or multiply numbers, they follow rules similar to ours, except that there are *no carries* into other digit positions.[2] Addition and multiplication of single-digit numbers are performed by a process that we would call "reduction mod 10." Any carry digits are simply ignored. So $9 \,\boldsymbol{+}\, 4 = 3$, $5 \,\boldsymbol{+}\, 5 = 0$, $9 \,\boldsymbol{\times}\, 4 = 6$, $5 \,\boldsymbol{\times}\, 4 = 0$, and so on.

Adding or multiplying larger numbers also follows the familiar procedures, but again always with the proviso that there are no carries. For example, summing 785 and 376 produces 51:

$$\begin{array}{r} 7\ 8\ 5 \\ \boldsymbol{+}\,3\ 7\ 6 \\ \hline 0\ 5\ 1 \\ \hline \end{array}$$

[1] This is the first of a proposed series of articles that will study various kinds of carryless arithmetic. Another installment will be devoted to "dismal arithmetic," in which operations on single digits are defined by $a \oplus b = \max\{a, b\}$, $a \otimes b = \min\{a, b\}$.

[2] Sociologists explain this by noting that the Carryless Islands were originally penal colonies, and, as penal institutions are generally known to have excellent dental care, the islanders were, happily, generally free of carries.

and the product of 643 and 59 is 417:

$$
\begin{array}{cccc}
 & 6 & 4 & 3 \\
\boldsymbol{\times} & & 5 & 9 \\
\hline
 & 4 & 6 & 7 \\
0 & 0 & 5 & \\
\hline
0 & 4 & 1 & 7 \\
\hline
\end{array}
$$

This article explores what elementary number theory would look like on these islands—what the analogs of the even numbers, the squares, primes, etc., would be in carryless arithmetic mod 10. We also explain how subtraction would work, although this is not a concept known to the islanders.[3]

3. Some initial calculations

We will use $\boldsymbol{+}$ and $\boldsymbol{\times}$ for addition and multiplication mod 10 in carryless arithmetic,[4] and $+$ and $\times$ for the standard operations used by the rest of the world.

Let's start with the carryless squares $n \boldsymbol{\times} n$. For $n = 0, 1, 2, 3$ we get 0, 1, 4, 9, Then for $n > 3$ we have $4 \boldsymbol{\times} 4 = 6$, $5 \boldsymbol{\times} 5 = 5$, $6 \boldsymbol{\times} 6 = 6$, $7 \boldsymbol{\times} 7 = 9$, $8 \boldsymbol{\times} 8 = 4$, $9 \boldsymbol{\times} 9 = 1$, $10 \boldsymbol{\times} 10 = 100, \ldots$, giving the sequence

$$0,\ 1,\ 4,\ 9,\ 6,\ 5,\ 6,\ 9,\ 4,\ 1,\ 100,\ 121,\ 144,\ 169,\ 186,\ 105,\ 126,\ 149,\ 164,\ \ldots .$$

It turns out that this is in the OEIS [17]. It is entry A059729, contributed by Henry Bottomley on February 20, 2001, although without any reference to earlier work on these numbers. Henry

[3] As mentioned in §1, carryless *addition* is Nim-addition generalized from base 2 to base 10. Note however that Nim-multiplication ([2, Chap. 14], [4, Chap. 6]) is completely different from the base-2 analog of our carryless multiplication. The algebraic approach given below in §5 shows that the full base-2 analog of our carryless arithmetic is the study of polynomials in $R_2[X]$. The OEIS contains a number of sequences related to such polynomials: A000695, A014580, A048720, ….

[4] We prefer not to use outlandish symbols such as ♣ and ♠, since $\boldsymbol{+}$ and $\boldsymbol{\times}$ are perfectly reasonable operations, although to our eyes they have rather strange properties. As Marcia Ascher remarks, writing about mathematics in indigenous cultures, "in many cases these cultures and their ideas were unknown beyond their own boundaries, or misunderstood when first encountered by outsiders" [1].

Bottomley also contributed a sequence (A059692) giving the carryless multiplication table, as well as several other sequences related to carryless products. Likewise the sequence of values of $n \boldsymbol{+} n$,

$$0,\ 2,\ 4,\ 6,\ 8,\ 0,\ 2,\ 4,\ 6,\ 8,\ 20,\ 22,\ 24,\ 26,\ 28,\ 20,\ 22,\ 24,\ 26,\ 28,\ 40,\ 42,\ \ldots\ ,$$

is entry A004520, submitted to the OEIS by one of the present authors around 1996, again with no references. (If these numbers are sorted and duplicates removed, we get the carryless "evenish" numbers, all of whose digits are even—see §8 and A014263.) Carryless arithmetic must surely have been studied before now, but the absence of references in [17] suggests that it is not mentioned in any of the standard texts on number theory. This motivated us to write this article, in which we will develop more of the mathematics behind such arithmetic.

4. The carryless primes

What about the carryless analog of prime numbers? If we require that a prime π is a number whose only factorization is 1 times itself, we are out of luck, since every number is divisible by 9, and there would be no primes at all. (For $9 \boldsymbol{\times} 1 = 9$, $9 \boldsymbol{\times} 2 = 8$, $9 \boldsymbol{\times} 3 = 7, \ldots$, $9 \boldsymbol{\times} 9 = 1$. So if we construct a number ρ by replacing all the 1's in π by 9's, all the 2's by 8's, ... then $\pi = 9 \boldsymbol{\times} \rho$, and π would not be a prime.)

However, there *are* primes, when defined in the right way. Since $1 \boldsymbol{\times} 1 = 1$, $3 \boldsymbol{\times} 7 = 1$ and $9 \boldsymbol{\times} 9 = 1$, all of $1, 3, 7$ and 9 divide 1 and so divide any number. We call $1, 3, 7$ and 9 the *units*, since that is the usual name for integers that divide 1. Units should not be counted as factors when considering if a number is prime (just as factors of -1 are ignored in ordinary arithmetic: $7 = (-1) \times (-7)$ doesn't count as a factorization when considering if 7 is a prime).

So we define a carryless prime to be a non-unit π whose only factorizations are of the form

$\pi = u \times \rho$ where u is a unit. Some computer experiments suggest that the first few primes are

$$21, 23, 25, 27, 29, 41, 43, 45, 47, 49, 51, 52, 53, 54, 56, 57, 58, 59, 61, 63, \ldots .$$

But there are surprising omissions in this list, resulting from some strange factorizations: $2 = 2 \times 51$, $10 = 56 \times 65$, $11 = 51 \times 61$. It is hard to be sure at this stage that the above list is correct, since there exist factorizations where one of the numbers is much larger than the number being factored, such as $2 = 4 \times 5005505553$. One property that makes carryless arithmetic interesting is the presence of *zero-divisors*: the product of two numbers can be zero without either of them being zero: $2 \times 5 = 0$, $628 \times 55 = 0$ (see §8). Perhaps 21 is the product of two really huge numbers? Nonetheless, the list *is* correct, as we will see (it is now sequence A169887 in [17]).

5. Algebra to the rescue

The secret to understanding carryless arithmetic is to introduce a little algebra.[5] Let R_{10} denote the ring of integers mod 10, and $R_{10}[X]$ the ring of polynomials in X with coefficients in R_{10}. Then we can represent carryless numbers by elements of $R_{10}[X]$: 21 corresponds to $2X+1$, 109 to X^2+9, and so on. Carryless addition and multiplication are simply addition and multiplication in the ring $R_{10}[X]$: our first example,

$$785 + 376 = 51 .$$

corresponds to

$$(7X^2 + 8X + 5) + (3X^2 + 7X + 6) = 5X + 1 ,$$

— the polynomials are added or multiplied in the usual way, with the coefficients then reduced mod 10. Conversely, any element of $R_{10}[X]$ represents a unique carryless number (just set $X = 10$

[5] Zariski and Samuel [18] is an excellent reference.

in the polynomial). In fact arithmetic in $R_{10}[X]$ is clearly exactly the same[6] as the arithmetic of carryless numbers.

Since $R_{10}[X]$ is a ring, we can not only add and multiply, we can also subtract, something the Carryless Islanders never considered. The negatives of the elements of R_{10} are $-1 = 9$, $-2 = 8, \ldots$, $-9 = 1$, and similarly for the elements of $R_{10}[X]$. So the negative of a carryless number is its "10's complement," obtained by replacing each nonzero digit d by $10 - d$, for example $\boldsymbol{-}702 = 308$. To subtract A from B, we add $\boldsymbol{-}A$ to B: $650 \boldsymbol{-} 702 = 650 \boldsymbol{+} 308 = 958$. This is equivalent to doing elementary school subtraction where we can "borrow" but don't have to pay back!

The units in $R_{10}[X]$, that is, the elements that divide 1, are the constants $1, 3, 7, 9$, just as in Section 4, and the carryless primes that we defined there are the *irreducible* elements in $R_{10}[X]$, that is, non-units $f_{10}(X) \in R_{10}[X]$ whose only factorizations are of the form $f_{10}(X) = u\, g_{10}(X)$, where u is a unit and $g_{10}(X) \in R_{10}[X]$. The units can also be written as 1, $\boldsymbol{-}1, 3$ and $\boldsymbol{-}3$, which more closely relates them to the units 1 and -1 in ordinary arithmetic (3 and $\boldsymbol{-}3$ act in some ways like the imaginary units i and $-i$, squaring to -1, for example).

The key to further progress is to notice that R_{10}, the ring of integers mod 10, is the direct sum of the ring R_2 of integers mod 2 and the ring R_5 of integers mod 5. Given $r_{10} \in R_{10}$, we read it mod 2 and mod 5 to obtain a pair $[r_2, r_5]$ with $r_2 \in R_2$, $r_5 \in R_5$. The elements $0, 1, \ldots, 9 \in R_{10}$ (or equivalently the carryless digits $0, 1, \ldots, 9$) and their corresponding pairs $[r_2, r_5]$ are given by the following table:[7]

$$
\begin{array}{cccccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
[0,0] & [1,1] & [0,2] & [1,3] & [0,4] & [1,0] & [0,1] & [1,2] & [0,3] & [1,4]
\end{array}
\tag{1}
$$

[6]This can be used as a formal definition of carryless arithmetic mod 10, if the definition in §2 was felt to be too vague. It also shows that this arithmetic is commutative, associative and distributive.

[7]The Chinese Remainder Theorem guarantees that this is a one-to-one correspondence.

As a check, we note that $\{1\}$ is the (singleton) set of units in R_2, while $\{1,2,3,4\}$ is the set of units in R_5, so the pairs $[1,1]$, $[1,2]$, $[1,3]$ and $[1,4]$ correspondingly produce the units 1, 7, 3 and 9 of R_{10}.

Similarly, polynomials $f_{10}(X) \in R_{10}[X]$ correspond to pairs of polynomials $[f_2(X), f_5(X)]$, obtained by reading $f_{10}(X)$ respectively mod 2 and mod 5. Conversely, given any such pair of polynomials $[f_2(X), f_5(X)]$, there is a unique $f_{10}(X) \in R_{10}[X]$ that corresponds to them, which can be found using (1). We indicate this by writing $f_{10}(X) \leftrightarrow [f_2(X), f_5(X)]$.

If also $g_{10}(X) \leftrightarrow [g_2(X), g_5(X)]$, then $f_{10}(X) + g_{10}(X) \leftrightarrow [f_2(X) + g_2(X), f_5(X) + g_5(X)]$ and $f_{10}(X)g_{10}(X) \leftrightarrow [f_2(X)g_2(X), f_5(X)g_5(X)]$.

We are now in a position to answer many questions about carryless arithmetic.

6. The carryless primes, again

What are the irreducible elements $f_{10}(X) \in R_{10}[X]$? If $f_{10}(X) \leftrightarrow [f_2(X), f_5(X)]$ is irreducible then certainly f_2 and f_5 must be either units or irreducible (for if $f_2 = g_2 h_2$ then we have the factorization $[f_2, f_5] = [g_2, f_5][h_2, 1]$). Also $[f_2, f_5] = [f_2, 1][1, f_5]$, so one of f_2, f_5 must be irreducible and the other must be a unit. So the irreducible elements[8] in $R_{10}[X]$ consist of elements of the form $[f_2(X), u]$, where $f_2(X)$ is an irreducible polynomial mod 2 of degree ≥ 1 and $u \in \{1,2,3,4\}$, together with elements of the form $[1, f_5(X)]$, where $f_5(X)$ is an irreducible polynomial mod 5 of degree ≥ 1.

The irreducible polynomials mod 2 are X, $X+1$, $X^2+X+1, \ldots$, and the irreducible polynomials mod 5 are uX, $uX+v, \ldots$, where $u, v \in \{1,2,3,4\}$ (see entries A058943, A058945 in [17]). The first few irreducible elements in $R_{10}[X]$ are therefore $[X,1]$, $[X,2]$, $[X,3]$, $[X,4]$, $[X+1,1]$, $[X+1,2], \ldots$, and $[1,X]$, $[1,2X]$, $[1,3X]$, $[1,4X]$, $[1,X+1]$, $[1,2X+1], \ldots$. The corresponding carryless primes

[8] The *prime ideals* in $R_{10}[X]$, as distinct from the irreducible elements, all have a single generator, which is one of $[0,1], [1,0], [1,1], [f_2(X), 1], [1, f_5(X)]$, where $f_2(X)$, $f_5(X)$ are irreducible (cf. [18, Chap. III, Thm. 30]).

can be written down using (1). They are $56, 52, 58, 54, 51, 57, \ldots$, and $65, 25, 85, 45, 61, 21, \ldots$, and we can verify that the list in §4 is correct.

We will call a number with at least two digits in which all digits except the rightmost are even and the rightmost is odd an *e-type number* (A143712), and a number with at least two digits in which all digits except the rightmost are 0 or 5 and the rightmost is neither 0 nor 5 an *f-type number* (A144162). Similarly, we call the primes corresponding to the irreducible elements $[1, f_5(X)]$ *e-type primes*, and the primes corresponding to the irreducible elements $[f_2(X), u]$ *f-type primes*.

We also see that our concern about the primality of 21 in §3 was groundless. It is impossible for the length (in decimal digits) of a nonzero carryless product to be less than the length of both of the factors—this follows from the fact that if $\ell(n)$ is the number of decimal digits in the number $n > 0$ corresponding to a pair $[f_2(X), f_5(X)]$, then $\ell(n) = 1 + \max\{\deg f_2, \deg f_5\}$. So if $mn > 0$, $\ell(mn) \geq \min\{\ell(m), \ell(n)\}$.

Also, since we know how many irreducible polynomials mod 2 and mod 5 there are of given degree (see A001037, A001692 in [17]), we can write down a formula for the number of k-digit carryless primes, something that we cannot do for ordinary primes:

$$\frac{4}{k-1} \sum_{d \,\mathrm{divides}\, k-1} \mu\left(\frac{k-1}{d}\right)(2^d + 5^d),$$

for $k \geq 2$, where μ is the Möbius function (A008683). There are 28 primes with two digits (the twenty listed in §4, together with $65, 67, 69, 81, 83, 85, 87, 89$), 44 with three digits, … (A169962). For large k the number is about $4 \cdot 5^{k-1}/(k-1)$, whereas the number of ordinary primes with exactly k digits is much larger, about $9 \cdot 10^{k-1}/(k \log 10)$, so carryless primes are much rarer than ordinary primes.

7. The carryless squares, again

Squaring a mod 2 polynomial is easy: $f_2(X)^2 = f_2(X^2)$. So if n corresponds to the pair $[f_2(X), f_5(X)]$, n^2 corresponds to $[f_2(X^2), f_5(X)^2] = [f_2(X^2), 0] + [0, f_5(X)^2]$. This gives a two-step recipe for producing all the carryless squares. First find (using (1)) the carryless number m corresponding to $[0, f_5(X)^2]$, where $f_5(X)$ is any polynomial mod 5. The effect of adding a nonzero $[f_2(X^2), 0]$ changes some subset of the digits in positions $0, 2, 4, \ldots$ of m by the addition of 5 mod 10.

For example, if $f_5(X) = X+2$, $f_5(X)^2 = X^2+4X+4$, and by (1) $[0, f_5(X)^2]$ corresponds to the carryless square $m = 644$. We now add 5 mod 10 to any subset of the digits in positions $0, 2, 4, 6, \ldots$ of m (considering m extended by prefixing it with any number of zeros), obtaining infinitely many squares $644, 649, 144, 149, \ldots, 50644, 5050649, \ldots$.

This also leads to a formula for the number of k-digit carryless squares. For even k the number is 0, and for odd k it is

$$\frac{1}{2}\, 9 \cdot 10^{(k-1)/2} \ + \ 2^{(k-3)/2}$$

(zero is excluded from the count). There are five squares of length 1 (namely $1, 4, 5, 6$ and 9), 46 of length 3, … (see A059729, A169889, A169963). For large odd k there are about twice as many k-digit carryless squares as ordinary squares.

8. Divisors and factorizations

In this section we consider some questions related to divisors and the factorization of numbers into the product of carryless primes. Unfortunately, the existence of zero-divisors complicates matters, and it turns out that there is no natural way to define, for example, an analog of the usual sum-of-divisors function $\sigma(n)$.

In our analysis we define several classes of carryless numbers:

$\mathcal{U} := \{1, 3, 7, 9\}$, the units

$\mathcal{E} = \{0, 2, 4, 6, 8, 20, 22, \ldots\}$, the "evenish" numbers, in which all digits are even (A014263)

$\mathcal{F} = \{0, 5, 50, 55, \ldots\}$, the "fiveish" numbers, in which all digits are 0 or 5 (A169964)

$\mathcal{Z} := \mathcal{E} \cup \mathcal{F} = \{0, 2, 4, 5, 6, 8, 20, 22, \ldots\}$, the zero-divisors (A169884)

$\mathcal{N} = \{1, 3, 7, 9, 10, 11, 12, 13, \ldots\}$, the positive numbers not in $\mathcal{Z}$ (A169968)

Suppose d is a carryless divisor of n, that is, there is a number q such that $d \times q = n$. What can be said about the possible choices for q? One can show—we omit the straightforward proofs in this section—that

– if $d \in \mathcal{N}$ then there is a unique q

– if $d \in \mathcal{E}$ then $d \times q' = n$ if and only if $q' = q + v$ for some $v \in \mathcal{F}$

– if $d \in \mathcal{F}$ then $d \times q' = n$ if and only if $q' = q + e$ for some $e \in \mathcal{E}$

The same distinctions are needed when we describe factorizations into primes:

– if $n \in \mathcal{N}$ then n has a unique factorization as a carryless product of primes, up to multiplication by units[9]

– if $n \in \mathcal{E}$ then n has a unique factorization as 2 times a product of e-type primes, up to multiplication by units (in this case, every f-type prime divides n)

– if $n \in \mathcal{F}$ then n has a unique factorization as 5 times a product of f-type primes, up to multiplication by units (in this case, every e-type prime divides n)

The following examples illustrate the three types of factorization:

[9] It follows that any non-unit in $\mathcal{N}$ can be written both as $e \times f$ and $e' + f'$, where e and e' are e-type numbers and f and f' are f-type numbers.

$n \in \mathcal{N}$: we already saw $10 = 56 ✖ 65$. But we also have $10 = (3 ✖ 56) ✖ (7 ✖ 65) = 58 ✖ 25 = (9 ✖ 56) ✖ (9 ✖ 65) = 54 ✖ 45 = 9 ✖ 52 ✖ 25$, etc., illustrating the nonuniqueness. Also $11 = 51 ✖ 61$; $101 = 21 ✖ 29 ✖ 51$, $1234 = 23 ✖ 23 ✖ 23 ✖ 51 ✖ 51 ✖ 52$

$n \in \mathcal{E}$: $20 = 2 ✖ 65$, $22 = 2 ✖ 61$, $2468 = 2 ✖ 69 ✖ 69 ✖ 69$

$n \in \mathcal{F}$: $50 = 5 ✖ 52$, $505 = 5 ✖ 51 ✖ 51$

Here are the analogous statements about divisors:

– if $n \in \mathcal{N}$, n has only finitely many divisors. If d divides n and $u \in \mathcal{U}$, then $d ✖ u$ divides n. The divisors may be grouped into equivalence classes $d ✖ \mathcal{U}$. Since the sum of the elements of $\mathcal{U}$ is zero, so is the sum of the divisors of n.

– if $n \in \mathcal{E}$, d divides n, $u \in \mathcal{U}$ and $v \in \mathcal{F}$, then $d ✖ u ✚ v$ divides n. So n has infinitely many divisors, belonging to equivalence classes $d ✖ \mathcal{U} ✚ \mathcal{F}$.

– if $n \in \mathcal{F}$, d divides n, $u \in \mathcal{U}$ and $e \in \mathcal{E}$, then $d ✖ u ✚ e$ divides n. So n has infinitely many divisors, belonging to equivalence classes $d ✖ \mathcal{U} ✚ \mathcal{E}$.

Any attempt to define a sum-of-divisors function must specify how to choose representatives from the equivalence classes. There seems to be no natural way to do this. One possibility would be to choose the smallest decimal number in each class, but this seems unsatisfactory (since it depends on the ordering of decimal numbers, another concept the islanders seem not to be familiar with).

9. Summary and further number theory

In summary, we can help the Carryless Islanders by defining subtraction, prime numbers, and factorization into primes. But further concepts such as the number of divisors, the sum of divisors and perfect numbers seem to lie beyond these Islands.

However, many other carryless analogs are well-defined, including including triangular numbers (A169890), cubes (A169885), partitions (A169973), greatest common divisors and least common multiples, and so on. Some seem exotic, while other familiar sequences simply become periodic.[10]

We might also generalize beyond simple squares, cubes, etc. and investigate the properties of polynomials or power series based on carryless operations–How do these factor? What are their fixed points?–and so on.

Taking a different tack, carryless mod 10 partitions are enumerated in A169973, which may be derived as the coefficients of z^n in the formal expansion of the analog of the classic partition generating function $\prod_{k=1}^{\infty}(1+z^k)$, wherein powers of z are multiplied together by combining their exponents with carryless mod 10 addition instead of the ordinary sum.

There's a great deal yet to be explored in these Carryless Islands!

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[10]For example, the analog of the Fibonacci numbers coincides with the sequence of Fibonacci numbers read mod 10, A003893, which becomes periodic with period 60 (the periodicity of the Fibonacci numbers to any modulus being a well-studied subject, see sequence A001175). Similarly, the analogue of the powers of 2 (A000689) becomes periodic with period 4.

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